CHANGES IN GROUND WATER LEVEL IN IRRIGATED TERRAIN

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Periodic solutions of a second-order partial differential equation with the right-hand part nonlinearly dependent on the unknown function, or the function incorporated in the boundary conditions, are discussed in the literature [1-8]. The time t is an implicit variable in the expression of this nonlinear function.

Periodic variation of the level of ground water in irrigated terrain is treated as a problem of periodic variation over a semi-infinite region [8] with attention given to evaporation, which is reduced to a linear integral equation whose solution is found in limiting cases.

The solution of the problem is found below in a semi-infinite region and in a finite region.

1. We shall assume that the ground waters occupy a region $0<x<l$ between channels with respective water levels H_1 and H_2. At the point $x=x^\circ$ $(0<x^\circ<l)$, the ground water level h is measured. When that level reaches a height h_*, irrigation carried out at an intensity mc (where m is the porosity) ceases and then resumes, as h declines to the level h_{**} (evaporation is taken into account in this treatment). The rate of evaporation is md.

This problem reduces to one of finding the solution of the heat conduction equation with the right-hand part dependent in a relay (on-off) pattern on the ground water level at the point x°, and with the boundary conditions

$$h(0,\ t)=H_1,\qquad h(l,\ t)=H_2 \tag{1.1}$$

Assuming

$$h(x,\ t)=H_1+(H_2-H_1)\,xl^{-1}+u(x,\ t) \tag{1.2}$$

we now reduce the problem to one of finding the solution u(x, t) of the equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}+F\,[u\,(x^\circ,t)] \tag{1.3}$$

where we introduce the notation

$$F\,[u\,(x^\circ,t)]=\begin{cases} c \quad \text{for} & u\,(x^\circ,t)<u_* \\ & (u_{**}<u_*,\,c>0,\,d>0) \\ -d \quad \text{for} & u\,(x^\circ,t)>u_{**}\end{cases} \tag{1.4}$$

with the conditions

$$u(0,\ t)=0,\qquad u(l,\ t)=0 \tag{1.5}$$

Here we introduce the notation

$$u_*=h_*-H_1-(H_2-H_1)\,x^\circ l^{-1},\quad u_{**}=h_{**}-H_1-(H_2-H_1)\,x^\circ l^{-1} \tag{1.6}$$

We begin by considering this problem in a finite interval.

2. Now let $u(x^\circ,\ T_1)=u_*$ when $t=T_1$, and $u(x^\circ,\ T)=u_{**}$ when $t=T$. Then $u(x,\ T)=u(x,\ 0)$, where T is the period of the oscillations. When $0\leq t\leq T_1$, $u(x,\ t)=u_1(x,\ t)$, and when $T_1\leq t\leq T$ $u(x,\ t)=u_2(x,\ t)$, and

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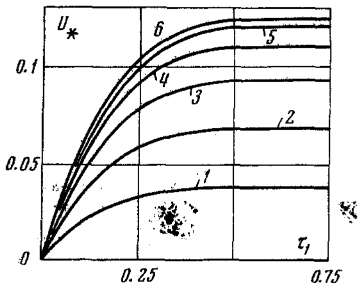

Fig. 1

the functions $u_1(x, t)$ and $u_2(x, t)$ satisfy the conditions

$$\frac{\partial u_1}{\partial t} = a^2 \frac{\partial^2 u_1}{\partial x^2} + c \tag{2.1}$$

$$u_1(0, t) = u_1(l, t) = 0 \tag{2.2}$$

$$\frac{\partial u_2}{\partial t} = a^2 \frac{\partial^2 u_2}{\partial x^2} - d \tag{2.3}$$

$$u_2(0, t) = u_2(l, t) = 0 \tag{2.4}$$

respectively.

We shall seek the solution of Eq. (2.1) in the form of the series

$$u_1(x, t) = \sum_{k=1}^{\infty} f_k(t) \sin \frac{\pi k x}{l} \tag{2.5}$$

Substitution of Eq. (2.5) into Eq. (2.1) yields an expression for the functions $f_k(t)$ $(k=1, 2, \ldots)$

$$f_k(t) = C_k \exp(-\lambda_k^2 t) - \frac{2cl^2}{a^2\pi^3 k^3}[(-1)^k - 1](1 - \exp(-\lambda_k^2 t)) \left(\lambda_k = \frac{\pi a k}{l}\right) \tag{2.6}$$

where C_k are unknown constants.

We now state the expressions for the functions $u_1(x, t)$ and $u_2(x, t)$

$$u_1(x, t) = \sum_{n=1}^{\infty} [C_n \exp(-\lambda_n^2 t) - \frac{2cl^2}{a^2\pi^3 n^3}[(-1)^n - 1](1 - \exp(-\lambda_n^2 t))] \sin \frac{\pi n x}{l} \qquad (0 \leqslant t \leqslant T_1)$$

$$u_2(x, t) = \sum_{n=1}^{\infty} \left\{D_n \exp[-\lambda_n^2 (t - T_1)] + \frac{2dl^2}{a^2\pi^3 n^3}[(-1)^n - 1](1 - \exp[-\lambda_n^2 (t - T_1)])\right\} \sin \frac{\pi n x}{l} \qquad (T_1 \leqslant t \leqslant T) \tag{2.7}$$

For brevity, we introduce the notation

$$\alpha_n = \frac{2l^2[(-1)^n - 1]}{a^2\pi^3 n^3}, \qquad \beta_n = \exp(-\lambda_n^2 T_1)$$
$$\gamma_n = \exp(-\lambda_n^2 (T - T_1)), \qquad \delta_n = \exp(-\lambda_n^2 T) \tag{2.8}$$

Clearly, the equations

$$u_1(x, 0) = u_2(x, T), \qquad u_1(x, T_1) = u_2(x, T_1) \tag{2.9}$$

must be satisfied.

In virtue of Eqs. (2.7), (2.8), we derive from Eqs. (2.9) some expressions for the constants C_n and D_n

$$C_n = \frac{\alpha_n\{-c(\gamma_n - \delta_n) + d(1 - \gamma_n)\}}{1 - \delta_n}, \quad D_n = \frac{\alpha_n\{-c(1 - \beta_n) + d(\beta_n - \delta_n)\}}{1 - \delta_n} \tag{2.10}$$

It is clear from Eqs. (2.8) and (2.10) that the series (2.7) converge uniformly.

The values of T_1, T are found as the smallest roots of the equations $u(x°, T_1) = u_*$ and $u(x°, T) = u_{**}$. In virtue of Eqs. (2.7) and (2.10),

$$\sum_{n=1}^{\infty} \frac{\alpha_n\{-c(1 - \beta_n) + d(\beta_n - \delta_n)\}}{1 - \delta_n} \sin \frac{\pi n x°}{l} = u_*$$
$$\sum_{n=1}^{\infty} \frac{\alpha_n\{-c(\gamma_n - \delta_n) + d(1 - \gamma_n)\}}{1 - \delta_n} \sin \frac{\pi n x°}{l} = u_{**} \tag{2.11}$$

We now consider the case $c=d$, $T=2T_1$. Equations (2.10) and (2.11) become

$$C_n = -D_n = c\alpha_n \frac{1 - \beta_n}{1 + \beta_n} \tag{2.12}$$
$$-\sum_{n=1}^{\infty} c\alpha_n \frac{1 - \beta_n}{1 + \beta_n} \sin \frac{\pi n x°}{l} = u_* = -u_{**}$$

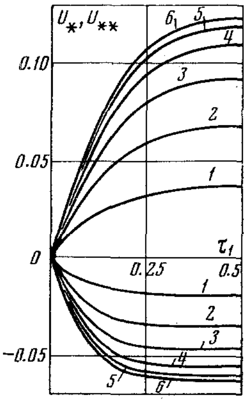

Fig. 2

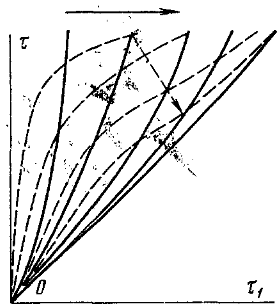

Fig. 3

Now let $x^\circ = l/2$. Then in the light of Eqs. (2.8), we find from Eq. (2.12)

$$C_{2k-1} = -\frac{4l^2}{a^2\pi^3}\frac{c}{(2k-1)^3}\frac{1-\beta_{2k-1}}{1+\beta_{2k-1}}, \qquad C_{2k} = 0$$

$$u_* = \frac{4l^2c}{a^2\pi^3}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^3}\frac{1-\beta_{2k-1}}{1+\beta_{2k-1}} \qquad (\beta_{2k-1} = \exp(-\lambda_{2k-1}{}^2 T_1)) \tag{2.13}$$

We now consider the function

$$y(T_1) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^3}\frac{1-\beta_{2k-1}(T_1)}{1+\beta_{2k-1}(T_1)}$$

Clearly, $y(0)=0$, $y'(T_1)>0$. As T_1 goes to infinity, y tends to the value [10]

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^3} = \frac{\pi^3}{32}$$

Hence, we arrive at the following result: a single root $T_1 > 0$ of Eq. (2.13) exists when the inequality

$$U_* = \frac{a^2u_*}{l^2c} < \frac{1}{8}$$

is satisfied.

Consequently, a unique solution of the problem exists under those conditions. It is described by Eqs. (2.7), where C_n and D_n are given by Eqs. (2.12).

If $c=d$, $T=2T_1$, then Eq. (2.12) acquires the form

$$U_* = \frac{a^2u_*}{l^2c} = \frac{4}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\frac{1-\beta_{2k-1}}{1+\beta_{2k-1}}\sin\frac{(2k-1)\pi x^\circ}{l} \tag{2.14}$$

It can be readily seen, from Eq. (2.14), that $U_*(0)=0$. With increasing $\tau_1 = a^2T_1/l^2$, as the calculations show, U_* increases, but as τ_1 goes to infinity U_* tends to

$$U_\infty = \frac{1}{2}\frac{x^\circ}{l}\left(1-\frac{x^\circ}{l}\right)$$

Accordingly, even in that case a unique solution τ_1 exists when $U_* < U_\infty$.

In the general case, when the notation

$$U_* = \frac{a^2u_*}{l^2c}, \quad U_{**} = \frac{a^2u_{**}}{l^2c}, \quad \tau_1 = \frac{a^2T_1}{l^2}, \quad \tau = \frac{a^2T}{l^2}, \quad \Delta = \frac{d}{c} \tag{2.15}$$

is introduced, we obtain from Eqs. (2.11) the following equations for the functions $U_*(\tau_1, \tau)$ and $U_{**}(\tau_1, \tau)$:

$$U_* = \frac{4}{\pi^3}\sum_{k=1}^{\infty}\frac{[1-\beta_{2k-1}-\Delta\beta_{2k-1}(1-\gamma_{2k-1})]}{(2k-1)^3(1-\beta_{2k-1}\gamma_{2k-1})}\sin\frac{(2k-1)\pi x^\circ}{l}$$

$$U_{**}=\frac{4}{\pi^3}\sum_{k=1}^{\infty}\frac{[\gamma_{2k-1}(1-\beta_{2k-1})-\Delta(1-\gamma_{2k-1})]}{(2k-1)^3(1-\beta_{2k-1}\gamma_{2k-1})}\sin\frac{(2k-1)\pi x^\circ}{l} \tag{2.16}$$

$$(\beta_{2k-1}=\exp[-\pi^2(2k-1)^2\tau_1],\quad \gamma_{2k-1}=\exp[-\pi^2(2k-1)^2(\tau-\tau_1)])$$

It is clear from Eqs. (2.16) that, as τ_1 goes to infinity, U_* tends to U_∞, and U_{**} tends to $-\Delta U_\infty$.

At large values of k, the k-th term of the series (2.16) does not exceed the k-th term of the series

$$S=\frac{4(1+\Delta)}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3} \tag{2.17}$$

The estimate of the residual term $R_n^{(i)}$ (i = 1, 2) of the series U_* and U_{**} used in the calculations was derived from Eq. (2.17):

$$|R_n^{(i)}|<\varepsilon,\quad n>N,\qquad E\left(\frac{N+1}{2}\right)\geqslant 100\sqrt{\frac{4}{3}\,\frac{(c+d)}{\pi^3}},$$

where E (x) is the integral part of the number x.

Figure 1 displays graphs of the function U_*, defined by Eq. (2.14), as a function of τ_1, for different values of the ratio x°/l.

U_* and U_{**} are plotted vs τ_1 in Fig. 2 for the case $\Delta=0.5$, $\tau=3\tau_1$.

Curves 1-6 correspond to the respective ratios x°/l = 0.08333, 0.91667; 0.16667, 0.83333; 0.25, 0.75; 0.33333, 0.66667; 0.41667, 0.58333; 0.5.

It is clear from the shape of the curves $U_*(\tau_1, b\tau_1)$, $U_{**}(\tau_1, b\tau_1)$, similar to those plotted in Fig. 1 and Fig. 2, that each pair of values of U_* and U_{**} such that $-\Delta U_\infty < U_* < U_\infty$, $-\Delta U_\infty < U_{**} < U_\infty$ corresponds to a single pair of values of τ_1, τ.

The shape of curves of constant values of U_* (continuous curves) and of U_{**} (broken curves) is shown in Fig. 3 in the τ_1, τ plane. The direction of increasing U_* and U_{**} is indicated by the arrows.

3. We consider the same problem in the semi-infinite region. The solution of the equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}+f(\tau) \tag{3.1}$$

with the boundary condition u (0, t) = 0 and without the initial conditions becomes [9]

$$u(x,t)=\frac{1}{a\sqrt{\pi}}\int_{-\infty}^{t}\frac{f(\tau)}{\sqrt{t-\tau}}\left(\int_{-\infty}^{\infty}\exp\left(-\frac{x^2+\xi^2}{4a^2(t-\tau)}\right)\operatorname{sh}\frac{x\xi}{2a^2(t-\tau)}\,d\xi\right)d\tau \tag{3.2}$$

Using the formula [10]

$$\int_0^{\infty}\exp\left(-\frac{x^2+\xi^2}{4a^2(t-\tau)}\right)\operatorname{sh}\frac{x\xi}{2a^2(t-\tau)}\,d\xi=a\sqrt{\pi(t-\tau)}\,\Phi\left(\frac{x}{2a\sqrt{t-\tau}}\right)$$

$$\left(\Phi(z)=\frac{2}{\sqrt{\pi}}\int_0^z\exp(-s^2)\,ds\right) \tag{3.3}$$

we restate the solution (3.2) of Eq. (3.1) in the following manner:

$$u(x,t)=\int_{-\infty}^{t}f(\tau)\,\Phi\left(\frac{x}{2a\sqrt{t-\tau}}\right)d\tau \tag{3.4}$$

We shall now assume that the function $f(\tau)$ takes on the values

$$f(\tau)=\begin{cases} c & \text{for}\quad kT\leqslant t\leqslant kT+T_1 \\ & \quad (k=0,\pm1,\pm2,\ldots) \\ -d & \text{for}\quad kT+T_1\leqslant t\leqslant (k+1)\,T \end{cases} \tag{3.5}$$

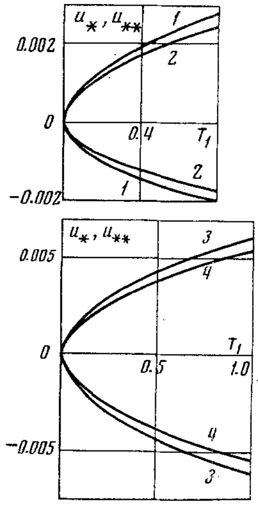

Fig. 4

Substitution of Eq. (3.5) into Eq. (3.4) yields the solution $u(x, t)$ in the form of a series, where we assume $u(x, t) = u_1(x, t)$ when $0 \le t \le T_1$, and $u(x, t) = u_2(x, t)$ when $T_1 \le t \le T$:

$$u_1(x, t) = c\int_0^t \Phi\left(\frac{x}{2a\sqrt{t-\tau}}\right) d\tau + S(x, t) \tag{3.6}$$

$$u_2(x, t) = -d\int_{T_1}^t \Phi\left(\frac{x}{2a\sqrt{t-\tau}}\right) d\tau + c\int_0^{T_1} \Phi\left(\frac{x}{2a\sqrt{t-\tau}}\right) d\tau + S(x, t)$$

$$S(x, t) = \sum_{k=0}^{-\infty}\left[-d\int_{(k-1)T+T_1}^{kT} \Phi\left(\frac{x}{2a\sqrt{t-\tau}}\right) d\tau + c\int_{(k-1)T}^{(k-1)T+T_1} \Phi\left(\frac{x}{2a\sqrt{t-\tau}}\right) d\tau\right]$$

Uniform convergence of the series $u_1(x, t)$ and $u_2(x, t)$, if $cT_1 = d(T-T_1)$ is inferred from the asymptotic representation of the probability integral $\Phi(z)$ at small values of z:

$$\Phi(z) = \frac{2}{\sqrt{\pi}} z + \ldots$$

and from Eqs. (3.6).

We can show, on the basis of the asymptotic representation of the probability integral at small values of z, that the series $\partial u_1(x, t)/\partial t$ and $\partial u_2(x, t)/\partial t$ converge uniformly and that, at all values of $x°$, the inequalities

$$\frac{\partial u_1(x°, t)}{\partial t} > 0, \qquad \frac{\partial u_2(x°, t)}{\partial t} < 0$$

hold, so that formulas (3.6) yield the solution of the unknown nonlinear problem. The constant T_1 is found from one of the equations

$$u_1(x°, T_1) = u_*, \qquad u_2(x°, T) = u_{**} \tag{3.7}$$

where $u(x, t)$ and $u_2(x, t)$ are defined by Eqs. (3.6). We readily see from Eqs. (3.6) and (3.7) that in the limit as $T_1 \to 0$, $u_* \to 0$ and $u_{**} \to 0$, and in the limit as $T \to \infty$, $u_* \to \infty$, and $u_{**} \to -\infty$.

At large values of n, the series terms are of the order of magnitude

$$a_n \approx An^{-5/2}, \quad b_n \approx An^{-5/2}, \quad A = -\frac{28cT_1z°^3}{3\sqrt{\pi}}\left(z° = \frac{x°}{2a}\right).$$

Figure 4 shows plots of u_*/c and u_{**}/c as functions of T_1 for the cases $d/c = 0.5$ (curve 1 corresponds to the value $z° = 0.0030$, curve 2 to the value $z° = 0.0027$) and $d/c = 1$ (curve 3 corresponds to the value $z° = 0.0030$, curve 4 to the value $z° = 0.0027$).

It is obvious from those graphs that each value of $u_* > 0$ corresponds to a unique T_1; but $u_{**} < 0$ is not arbitrary, being determined from the second of Eqs. (3.7).

4. Assuming a finite distance l between the channels, we now find the solution of the problem discussed in Sec. 2, with the initial conditions as stated, i.e., we find the solution of the problem (1.3)-(1.5) with the condition

$$u_1(x, 0) = \varphi_1(x) \tag{4.1}$$

where the function $\varphi_1(x)$ will be assumed to satisfy the Dirichlet conditions in the interval $0 \le x \le l$, and we shall show that under the restrictions imposed upon the function $\varphi_1(x)$ and on the constants figuring in the conditions of the problem, the solution of problem tends to the periodic solution found in section 2 as the time t increases without bound.

It is readily seen that the solution of the problem in question, if one exists, can be cast in the form of the series

$$u_1^{(i+1)}(x, t) = \sum_{n=1}^{\infty}\Big[A_n^{(i+1)} \exp\Big\{-\lambda_n^2\Big(t - \sum_{j=0}^{i} T^{(j)}\Big)\Big\} - c\alpha_n\Big\{1 - \exp\Big[-\lambda_n^2\Big(t - \sum_{j=0}^{i} T^{(j)}\Big)\Big]\Big\}\Big] \sin\frac{\pi n x}{l}$$

$$\left(\sum_{j=0}^{i} T^{(j)} \leqslant t \leqslant \sum_{j=0}^{i} T^{(j)} + T_1^{(i+1)}, \quad i = 0, 1, 2, 3, \ldots, T^{(0)} = 0 \right) \tag{4.2}$$

$$u_2^{(i+1)}(x, t) = \sum_{n=1}^{\infty} \Big[B_n^{(i+1)} \exp \Big\{ -\lambda_n^2 \Big(t - \sum_{j=0}^{i} T^{(j)} - T_1^{(i+1)} \Big) \Big\} \tag{4.3}$$

$$+ d\alpha_n \Big\{ 1 - \exp \Big[-\lambda_n^2 \Big(t - \sum_{j=0}^{i} T^{(j)} - T_1^{(i+1)} \Big) \Big] \Big\} \Big] \sin \frac{\pi n x}{l}$$

$$\left(\sum_{j=0}^{i} T^{(j)} + T_1^{(i+1)} \leqslant t \leqslant \sum_{j=0}^{i} T^{(j)} + T^{(i+1)}, \quad i = 0, 1, 2, 3, \ldots \right)$$

Here $A_n^{(1)}$ are the Fourier coefficients of the function $\varphi_1(\xi)$, $T_1^{(i)}$, and $T^{(i)}-T_1^{(i)}$ are the roots of the equations

$$u_1^{(i+1)} \Big(x^\circ, \sum_{j=0}^{i} T^{(j)} + T_1^{(i+1)} \Big) = u_*, \quad u_2^{(i+1)} \Big(x^\circ, \sum_{j=0}^{i} T^{(j)} + T^{(i+1)} \Big) = u_{**} \tag{4.4}$$

(u_* and u_{**} are given by Eqs. (2.15)-(2.16), which, in the light of Eqs. (4.2) and (4.3), are written as follows:

$$\sum_{n=1}^{\infty} B_n^{(i)} \sin \frac{\pi n x^\circ}{l} = u_*, \quad \sum_{n=1}^{\infty} A_n^{(i+1)} \sin \frac{\pi n x^\circ}{l} = u_{**} \qquad (i = 1, 2, 3, \ldots) \tag{4.5}$$

while the functions $B_n^{(i)}$ and $A_n^{(i+1)}$ $(i = 1, 2, 3, \ldots)$ are related by the equations

$$B_n^{(i)} = -c\alpha_n (1 - \beta_n^{(i)}) + \beta_n^{(i)} A_n^{(i)}, \quad A_n^{(i+1)} = d\alpha_n (1 - \gamma_n^{(i)}) + \gamma_n^{(i)} B_n^{(i)} \tag{4.6}$$

$$(\beta_n^{(i)} = \exp[-\lambda_n^2 T_1^{(i)}], \quad \gamma_n^{(i)} = \exp[-\lambda_n^2 (T^{(i)} - T_1^{(i)})])$$

It is also clear from Eqs. (4.2) and (4.3) that, in the limit as $t \to \infty$, $u_1^{(i+1)}(x^\circ, t) \to cU_\infty$,

$$u_2^{(i+1)}(x^\circ, t) \to -dU_\infty \qquad \left(U_\infty = \frac{x^\circ}{2a^2 l} \left(1 - \frac{x^\circ}{l} \right) \right)$$

We shall assume, as in paragraph 2, that the inequalities $-dU_\infty < u_* < cU_\infty$, $-dU_\infty < u_{**} < cU_\infty$, i.e., $-U_\infty < (u_* - u_{**})/(c+d) = (h_* - h_{**})/(c+d) < U_\infty$, are satisfied.

In order for the values of u_* and u_{**} to be attainable at finite instants of time, it will be sufficient that the inequalities

$$\varphi_1(x^\circ) < u_*, \quad \sum_{n=1}^{\infty} B_n^{(i)} \sin \frac{\pi n x^\circ}{l} = u_* > u_{**} \qquad (i = 1, 2, 3, \ldots)$$

$$\sum_{n=1}^{\infty} A_n^{(i)} \sin \frac{\pi n x^\circ}{l} = u_{**} < u_* \qquad (i = 2, 3, \ldots)$$

be satisfied.

Assuming $u_* - u_{**} > (c+d) pU_\infty$ (where p is some constant), we obtain $0 < \beta_n^{(j)} < 1$, $0 < \gamma_n^{(j)} < 1$ $(j = 1, 2, \ldots)$. Recalling that $\alpha_{2k} = 0$ $(k = 1, 2, 3, \ldots)$, we have

$$\beta_n^{(i)} = (\beta_1^{(i)})^{n^2}, \gamma_n^{(i)} = (\gamma_1^{(i)})^{n^2}$$

and it is clear from Eqs. (4.5), (4.6) that, when the values of i are sufficiently large, we can find the constants $T_1^{(i)}$ and $T^{(i)}-T_1^{(i)}$ in practice from the equations

$$\mathring{B}_1^{(i)} \sin \frac{\pi x^\circ}{l} + \mathring{B}_3^{(i)} \sin \frac{3\pi x^\circ}{l} = u_* - cU_\infty, \quad \mathring{A}_1^{(i)} \sin \frac{\pi x^\circ}{l} + \mathring{A}_3^{(i)} \sin \frac{3\pi x^\circ}{l}$$

$$= u_{**} + dU_\infty \qquad (B_n^{(i)} = -c\alpha_n + \mathring{B}_n^{(i)}, \quad A_n^{(i)} = d\alpha_n + \mathring{A}_n^{(i)}) \tag{4.7}$$

We find equations for $\beta_1^{(i)}$ and $\gamma_1^{(i)}$ from Eqs. (4.6) and (4.7)

$$\beta_1^{(i)} = \beta_1^{(0)} + A\beta_1^{(i)} [(\gamma_1^{(i-1)})^9 + (\beta_1^{(i)})^8], \quad \gamma_1^{(i)} = \gamma_1^{(0)} + B\gamma_1^{(i)} [(\beta_1^{(i)})^9 + (\gamma_1^{(i)})^8] \tag{4.8}$$

Here $\beta_1^{(0)}$, $\gamma_1^{(0)}$, A, B are constants, and the sign of A and B coincides with the sign of $\sin 3\pi x^\circ/l$. From the conditions $0 < \beta_1^{(0)} < 1$, $0 < \gamma_1^{(0)} < 1$, we have the inequalities:

$$a_1 > -\mu_1, \quad a_1 > \mu_2, \quad \mu_2 - \mu_1 < a_1 \qquad \left(\mu_1 = u_* - cU_\infty, \mu_2 = u_{**} + dU_\infty, \; a_i = -(c+d)\,\alpha_i \sin \frac{i\pi x^\circ}{l} \right)$$

If either $A < 0$, $B < 0$ or $0 < A < 1$, $0 < B < 1$, $\beta_1^{(0)} + 2A < 1$, $\gamma_1^{(0)} + 2B < 1$, and hence the inequalities

$$\mu_2 < a_1 + a_3, \quad \mu_1 > -a_1 - a_3, \quad \mu_2 - \mu_1 < a_1 + 2a_3$$

we can find $0 < \beta_1^{(i)} < 1$ and $0 < \gamma_1^{(i)} < 1$ from Eqs. (4.7), and then $\beta_1^{(i)}$ and $\gamma_1^{(i)}$ form monotonic sequences bounded above and below, so that β_1 and γ_1 have limits by the Weierstrass theorem. By substituting the values of β_1 and γ_1 into Eqs. (4.7) and (4.8), we see that these formulas practically coincide in that case with Eqs. (2.11), (2.8) for the periodic solution of the problem.

Consequently, when the above inequalities are valid, the solution of problem (1.3)-(1.5) with the initial condition (4.1) tends to the periodic solution (2.7), (2.8), (2.10) as the time t increases without bound.

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